ON THE VISCOUS BOUNDARY LAYER ON AN ELECTRODE IN A MEDIUM WITH VARIABLE ELECTRICAL CONDUCTIVITY

(O VIAZKOM POGRANICHNOM SLOE NA ELEKTRODE PRI PEREMENNOI ELEKTROPROVODNOSTI SREDY)

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In the paper [1] the formulation of the problem of the viscous boundary layer for constant electrical conductivity was examined. It was shown that, within the restrictions of boundary layer theory, the tangential component of the electric field does not change across the boundary layer, and the change in the normal component of electric field can be calculated from the condition j°_n = const .

Thus the problem of the boundary layer can be separated from that of the external flow.

If the temperature of the electrode is much lower than that of the external flow, then, due to the dependence of electrical conductivity on temperature, a space charge considerably greater than in the case σ = const may be concentrated in the boundary layer. Below are derived the equations which describe the change of the electric field across the boundary layer in this case. It is shown that if (due to large temperature difference between the flow core and the walls) the boundary layer resistance becomes comparable to the resistance of the flow core between the two electrodes, the boundary layer problem cannot, in general, be separated from the problem of the external flow. Estimates deduced from the boundary layer equations show that, for the latter case, there exists inside the viscous boundary layer a "thermal sublayer", in which an intense generation of Joule heat occurs and which carries the main electrical resistance. For simplicity, only the case of isotropic conductivity is investigated.

1. A viscous boundary layer on an electrode in a magnetic field which is parallel to it (this is the situation, as a rule, on electrodes in magnetohydrodynamic devices) carries a space electric charge [1]. The space charge density is determined by Ohm's law, and for $R_m \ll 1$ and $\sigma = \sigma(T) = \sigma(x,y,z)$ is given by relation

$$4\pi\rho_e = -\frac{1}{c}\mathbf{H}\,\mathrm{rot}\,\mathbf{v} - \frac{1}{\sigma^2}\,\mathbf{j}\cdot\mathrm{grad}\,\sigma \tag{1.1}$$

Making boundary layer approximations [1] for the hydrodynamic quantities in this equation, we find that the main part of the charge density in the boundary layer is determined by the relation

$$4\pi\rho_e^\circ = -\frac{1}{c}\frac{\partial}{\partial y}(\mathbf{v}_\tau \times \mathbf{H})_y - \frac{1}{\sigma^2}\,j_y^\circ\,\frac{\partial\sigma}{\partial y} \tag{1.2}$$

Here, y is the coordinate normal to the surface next to the flow, the subscript τ denotes the projection of a vector on the tangent plane x, z, the superscript $^\circ$ indicates that the corresponding quantities are evaluated with accuracy up to terms of order unity (with respect to δ, the boundary layer thickness).

For σ = const , the second term (B) in Equation (1.2) goes to zero, the charge density in the boundary layer is determined by the first term (A) in (1.2); its influence on the distribution of the electric field in the layer was studied in detail in [1]. The second term in (1.2) is connected with the charge density which occurs as a result of the variation of the conductivity. In view of the linearity of the equations of electrodynamics, the influence of this charge on the distribution of the field in the boundary layer may be studied separately. Besides, in a number of cases, which are mainly to be investigated here, the charge density connected with the variation of the conductivity (B) is significantly greater than the charge density connected with the variation of the velocity through the boundary layer (A). In fact, if the mean electrical conductivity in the boundary layer is denoted by σ_0, the ratio of the terms in the right-hand side of (1.2) is of order $B/A \sim j_y^0 c/\sigma_0 UH$. If, in addition,

$$j_y^\circ \gtrsim \frac{\sigma_\infty UH}{c} \qquad (1.3)$$

then

$$\frac{B}{A} \gtrsim \frac{\sigma_\infty}{\sigma_0} \sim \frac{r_0}{r}\frac{h}{\delta} \gg 1 \quad \text{for } r \sim r_0, \quad r = \frac{h}{\sigma_\infty}, \quad r_0 = \int_0^\delta \frac{dy}{\sigma} \sim \frac{\delta}{\sigma_0} \qquad (1.4)$$

Here, r is the internal resistance of the flow core between two electrodes (σ_∞ = const) and r_0 is the resistance of the boundary layer. In what follows, we shall be interested in the conditions for which $r_0 \gtrsim r$. This may occur if the temperatures of the flow core and the wall are very different. In fact,

$$r_0 = \int_0^\delta \frac{dy}{\sigma} \approx \int_{T_w}^{T_\infty} \left(\frac{\partial T}{\partial y}\right)^{-1} \frac{dT}{\sigma}$$

The derivative $\partial T/\partial y$ is bounded, while σ is a rapidly changing function of temperature; therefore, $r_0 \to \infty$ for $T_w \to 0$. Thus it is clear that, by cooling the wall, the condition

$$r_0 \gtrsim r \qquad (1.5)$$

can be fulfilled, as will be assumed in the following.

The condition (1.3) is fulfilled as a rule in magnetohydrodynamic generator and accelerator flows. For large external loads of an mhd generator, condition (1.3) may be violated. In these cases, notwithstanding the variation of the conductivity, the charge density in the boundary layer is determined by the first term A in (1.2), and all the conclusions of paper [1] are valid.

In the following, we shall be interested in a boundary layer on a "cold" electrode, for which relations (1.3) to (1.5) hold. Then the charge density in the boundary layer is given by the relation

$$4\pi\rho_e = -\frac{1}{\sigma^2} j_y{}^\circ \frac{\partial\sigma}{\partial y} \tag{1.6}$$

If there are no strong external electric fields parallel to the wall, then from Equations div $\mathbf{j} = 0$ and rot $\mathbf{E} = 0$ it follows (this is very easy to show by making use of the estimates in Section 2 of the given paper) that, within the boundary layer approximation, $\partial j_y{}^\circ / \partial y = 0$ and, therfore, $j_y{}^\circ = j_y{}^\circ(x, z)$ in Equation (1.6).

From Equation div $\mathbf{E} = 4\pi\rho_e$ and (1.6) we obtain, within the restrictions of boundary layer theory,

$$\frac{\partial E_y}{\partial y} = -\frac{1}{\sigma^2} j_y{}^\circ \frac{\partial\sigma}{\partial y} \tag{1.7}$$

From this, we obtain for the distribution of electric potential in the boundary layer

$$\varphi(x,y,z) = -j_y{}^\circ \int_0^y \frac{dy}{\sigma} + \varphi_w(x, z) \tag{1.8}$$

Here, $\varphi_w(x,z)$ is the distribution of potential over the electrode surface. If the electrode is continuous, then $\varphi_w = \text{const}$.

Let the electrode be continuous; then $\mathbf{E}_{\tau w} = 0$. From (1.8) we obtain the following relations for the distribution of the tangential component of the electric field across the boundary layer

$$\frac{\partial E_x}{\partial y} = -\frac{\partial^2\varphi}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{j_y{}^\circ}{\sigma}\right), \qquad E_{x\infty} = \frac{\partial}{\partial x} j_y{}^\circ \int_0^\delta \frac{dy}{\sigma} = \frac{\partial}{\partial x}(j_y{}^\circ r_0)$$
$$\frac{\partial E_z}{\partial y} = -\frac{\partial^2\varphi}{\partial y\,\partial z} = \frac{\partial}{\partial z}\left(\frac{j_y{}^\circ}{\sigma}\right), \qquad E_{z\infty} = \frac{\partial}{\partial z} j_y{}^\circ \int_0^\delta \frac{dy}{\sigma} = \frac{\partial}{\partial z}(j_y{}^\circ r_0) \tag{1.9}$$

From this it follows that in the flow core there is a tangential electric field comparable in magnitude with the normal component of the electric field. In fact, in view of (1.5) and (1.9),

$$\frac{E_{y\infty}}{E_{\tau\infty}} \sim \frac{L}{\sigma_\infty r_0} \lesssim \frac{L}{h} \tag{1.10}$$

It is evident, then, that the tangential component of the electric field must be included in the formulation of the problem in the flow core.

For $\sigma \approx \sigma_\infty$, the ratio $E_{y\infty} / E_{\tau\infty} \sim L / \delta \gg 1$, and, therefore, within the approximations of boundary layer theory, it may be assumed that the tangential component of the electric field does not change across the boundary layer [1]. The appearance of the tangential component is connected with the high charge concentration near the cold wall, which creates near the wall the high electrical fields that are required for flow of the given current density (1.3) across a boundary layer with low conductivity

$$\frac{E_{yw}}{E_{y\infty}} \sim \frac{\sigma_\infty}{\sigma_w} \gg 1$$

Thus, the relations for the change of electric field across the boundary layer differ from the conditions on a surface of discontinuity carrying an electric charge. This fact, which is unexpected, in view of the small thickness of the bounday layer, may be obtained from an investigation of Poisson's equation for the electric potential, if the right-hand side of Poisson's equation is given in accordance with (1.6).

Let us consider a strip of width δ in which the charge density is distributed according to (1.6). The distribution of electric field in the strip is then determined by Equation

$$\Delta\varphi = -4\pi\rho_e = -j_y^\circ \frac{\partial}{\partial y}\frac{1}{\sigma} \qquad (\varphi(x, 0) = 0) \tag{1.11}$$

From this, using Green's formula [2] for the half-strip, we obtain

$$\varphi(x_0, \delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\delta} \ln\frac{1}{\sqrt{(x-x_0)^2+(y-\delta)^2}}\, j_y^\circ(x)\frac{\partial}{\partial y}\frac{1}{\sigma}\,dx\,dy -$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x,\delta)\lim_{y\to\delta}\frac{y-\delta}{(x-x_0)^2+(y-\delta)^2}\,dx -$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\left(\frac{1}{x-x_0}\right)\frac{j_y^\circ(x)}{\sigma(x,\delta)}\,dx + \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\frac{1}{\sqrt{(x-x_0)^2+\delta^2}}\frac{j_y^\circ(x)}{\sigma(x,0)}\,dx$$

Or, integrating by parts in the first integral,

$$\varphi(x_0, \delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_0^{\delta}\frac{j^\circ_y(x)}{\sigma(x,y)}\frac{y-\delta}{(x-x_0)^2+(y-\delta)^2}\,dy -$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x,\delta)\lim_{y\to\delta}\frac{y-\delta}{(x-x_0)^2+(y-\delta)^2}\,dx \tag{1.12}$$

The last integral on the right-hand side of (1.12) is different from zero since the integrand has a singularity at the point ($x = x_0$, $y = \delta$), and is equal to $-\frac{1}{2}\varphi(x_0, \delta)$. Therefore,

$$\varphi(x_0,\delta) = \frac{1}{\pi}\int_{-\infty}^{\infty}dx\int_0^{\delta}\frac{j_y^\circ(x)}{\sigma(x,y)}\frac{y-\delta}{(x-x_0)^2+(y-\delta)^2}\,dy \tag{1.13}$$

The main contribution to the value of the integral in (1.13) comes from the neighborhood of $x = x_0$. Integration outside that neighborhood gives a result $\sim\delta$ (the function j_y°/σ is bounded) and, therefore, within boundary layer theory, these terms may be neglected. On integration with respect to x in the neighborhood of $x = x_0$, the function

$$j_y^\circ(x)\,/\,\sigma(x,y) \approx j_y^\circ(x_0)\,/\,\sigma(x_0,y).$$

Integrating with respect to x in (1.13), we obtain

$$\varphi(x_0,\delta) = -j_y^\circ(x_0)\int_0^{\delta}\frac{dy}{\sigma(x_0,y)} \tag{1.14}$$

This result follows from (1.8).

The result contained in Equations (1.8) and (1.14) was obtained on the basis of an analysis of Equations div $\mathbf{E} = 4\pi\rho_e$ and div $\mathbf{j} = 0$. At first glance, this result contradicts Equation rot $\mathbf{E} = 0$, since from this equation, for the plane case for example, it follows that

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \qquad (1.15)$$

In fact, if the characteristic length for the variation of E_x were δ , then, because of (1.10), it would follow from (1.15) that $\partial E_x/\partial y = 0$ in the boundary layer, rather than the result in (1.9). Actually, there is no contradiction. The charaateristic length for the variation of E_x is much smaller than the quantity δ and is comparable to the thickness of the cold sublayer, in which the main change in the electrical conductivity occurs and where E_y is very large. The relation between these quantities is such that the two terms in (1.15) are of the same order. The following section is devoted to an examination of this question.

2. From Equations (1.9) it follows that the characteristic length (we denote it by δ^*) over which $\mathbf{E}_\tau$ changes corresponds to the characteristic distance across the boundary layer in which the electrical conductivity σ changes from the value σ_w to the value σ_∞, since it is precisely the layer of thickness δ^* that determines the resistance r_0.

Since $\sigma = \sigma(p,T)$, it is clear that the quantity δ^* determines in some sense the thickness of the thermal boundary layer (or sublayer). Existing calculations for the magnetohydrodynamic boundary layer [3] show that, for a weakly ionized medium ($P \sim 1$) , there exists near the cold wall a region of significant heating of the gas, and the temperature profile has an essentially nonmonotonous character, with maximum near the wall. Below, it will be shown that the phenomenon is characteristic for the boundary layer on a cold electrode. Thus it is clear that the thickness of the thermal boundary layer δ^* , defined as the distance to the point nearest the wall with temperature $T \approx T_\infty$, will be smaller than the thickness of the viscous boundary layer δ . Comparing the viscous and electromagnetic terms in the equation of motion, we obtain, in view of (1.3),

$$\frac{\eta U}{\delta^2}\frac{c}{jH} \sim \frac{\eta U}{\delta^2}\frac{c^2}{UH^2\sigma_\infty} \sim \frac{L^2}{\delta^2 R}\frac{c^2 U\rho}{\sigma_\infty H^2 L} \sim \frac{L^2}{\delta^2 R}(mL)^{-1}\left(R = \frac{UL}{\nu}\right) \qquad (2.1)$$

From this it follows that the thickness of a boundary layer with substantial magnetohydrodynamic effect $(mL \sim 1)$ is determined, as in ordinary hydrodynamics, by the relation $\delta \sim L/\sqrt{R}$.

On the other hand, comparing viscous and electromagnetic terms in the energy equation, we obtain

$$\frac{\eta U^2}{\delta^2}\frac{\sigma^*}{j^2} \sim \frac{\eta U^2}{\delta^2}\frac{\sigma^* c^2}{\sigma^2_\infty U^2 H^2} \sim \frac{1}{mL}\frac{\sigma^*}{\sigma_\infty} \sim \frac{r}{r_0}\frac{\delta^*}{h} \qquad (2.2)$$

(where σ^* is the characteristic magnitude of the electrical conductivity in the cold layer, $\delta^*/\sigma^* \sim r_0$) . From this it follows that if $\sigma^* \sim \sigma_\infty$, then the viscous and electeromagnetic terms are comparable in magnitude. On the other hand, if the electrode is cold and (1.5) holds, then the main heating within the thermal layer (δ^*) is due to Joule heating. Thus the quantity δ^* can be determined by equating the electromagnetic term in the enegy equation with the term which determines the electrical conductivity. Since, for a cold wall, the change of temperature in the thermal layer is of order T_∞ , we obtain the following relation for determining δ^*:

$$k\frac{T_\infty}{\delta^{*2}} \sim \frac{j^2}{\sigma^*} \tag{2.3}$$

Using the relation $c_pT / U^2 \sim 1 / (\gamma - 1) M^2$ and (1.3), (1.5), the relation (2.3) may be rewritten in the form

$$\frac{\delta^{*2}}{L^2} \sim \frac{1}{(\gamma-1)\, M^2PRmL}\,\frac{\sigma^*}{\sigma_\infty} \lesssim \frac{1}{(\gamma-1)\, M^2PRmL}\,\frac{\delta^*}{h}$$

From this we obtain an estimate of the thermal layer thickness,

$$\frac{\delta^*}{L} \lesssim \frac{1}{(\gamma-1)\, M^2PRmL}\,\frac{L}{h} \tag{2.4}$$

The relation (2.4) shows that the thermal layer or the layer of high electrical resistance is much thinner than the viscous dynamical boundary layer ($\delta^* / \delta \sim R^{-1/2}$).

From the physical point of view, the generation of a thin layer of sharp temperature change near a cold wall is plausible, since it is connected with the existence of strong heat sources near the wall, which are due to the flow of current across a layer of cold gas. The presence of this layer is explained by the fact that near a cold electrode there is strong heating of the gas, leading to a nonmonotonous temperature profile which is steep near the wall.

It is easy to verify, on the basis of (2.4),(1.3) and (1.5) and the relation $\delta^*/\sigma^* \sim r_0$, that in Equation (1.15) the two terms have the same order of magnitude, as stated at the end of Section 1.

3. Because of the existence at a cold electrode of a thin layer with a concentration of large electric space charge, the estimates made in [1] for the magnitude of the electromagnetic force are not applicable in the given case.

From relation (1.10) and (1.5) it follows that

$$E_\tau \sim \frac{UH}{c}$$

and therefore the estimates made in [1] for the tangential component of the electromagnetic force are valid. Therefore, in projections of the momentum equations onto the tangent plane, the term $\rho_e\mathbf{E}$ in the expression for the force may be neglected, and these equations will have the same form as in [1].

The ratio of the normal component of force connected with the space charge $\rho_e E_n$ to the normal component of force acting on the current $\sigma^{-1}(\mathbf{j} \times \mathbf{H})_n$ in the cold layer is given, in view of (1.6) and (2.4), by the following form:

$$\frac{\rho_e E_n}{c^{-1}(\mathbf{j}\times\mathbf{H})_n} \sim \frac{\sigma_\infty}{\sigma^*}\frac{U}{\sigma^*\delta^*} \gtrsim \frac{U^2}{c^2}\left(\frac{h}{L}\right)^5 \frac{1}{R_m}\,[(\gamma-1)\,M^2PRmL]^3 \tag{3.1}$$

Therefore, for high Reynolds and Mach numbers, it could turn out that $\rho_e E_n \gg c^{-1}(\mathbf{j}\times\mathbf{H})_n$. Then the normal component of the momentum equation gives

$$\frac{\partial p}{\partial y} = \rho_e E_n \tag{3.2}$$

If $\rho_e E_n \sim c^{-1}(\mathbf{j}\times\mathbf{H})_n$ then, within boundary layer theory, it follows from (3.2) that $\partial p/\partial y = 0$. If $\rho_e E_n \gg c^{-1}(\mathbf{j}\times\mathbf{H})_n$, then the change of pressure across the cold layer may become significant

$$\frac{\Delta p}{p} \sim \frac{\Delta p}{\rho U^2} \lesssim \frac{U^2}{c^2}\frac{R^2}{R_m}\left(\frac{h}{L}\right)^4 (mL)^3\,[(\gamma-1)PM^2]^2 \tag{3.3}$$

It is evident that for $\rho_e E_n \sim c^{-1}(\mathbf{j}\times\mathbf{H})_n$ the change of pressure in the cold sublayer and in the entire boundary layer is small ($\Delta p \ll p$). For $\rho_e E_n \gg c^{-1}(\mathbf{j}\times\mathbf{H})_n$ and not too large R and M, we have $\Delta p/p \ll 1$. In these cases, $\partial p/\partial y = 0$ in the boundary layer equations. If $\Delta p \gtrsim p$, then equation (3.2) must be used in the system of boundary layer equations.

It is easy to prove that the convective current $(\rho_e \mathbf{v})$ in the expression for the current density may be neglected, since velocities are small in the cold layer.

Thus, for $\Delta p / p \ll 1$ the boundary layer equations for a cold electrode have the same form as in [1], with the only difference that the tangential component of the electric field is determined, not by the solution of the outer problem, but by Equations (1.9). For $\Delta p / p \gtrsim 1$ the system of equations is made more complicated by the variation of pressure across the boundary layer. In connection with this, we obtain from (3.2) and (1.6) the equation for the pressure distribution

$$p = p_\infty + \frac{1}{8\pi}\, j_y^{\circ 2}\left(\frac{1}{\sigma^2} - \frac{1}{\sigma_\infty^2}\right) \tag{3.4}$$

4. The preceding analysis shows that, in the formulation of the problem of the boundary layer at a cold electrode, in satisfying conditions (1.5) it is necessary to take into account that the electric field in the boundary layer is determined as a solution of the outer problem (in the flow core) as well as a solution of the boundary layer problem (it depends on the boundary layer resistance). On the other hand, the solution of the flow core problem depends on the distribution of parameters across the boundary layer, through the magnitude of the tangential component of the electric field. In connection with this, for $r \sim r_0$ the outer problem does not differ in principle from the boundary layer problem. Of course, the division of the flow between the boundary layer and the flow core simplifies the problem. Nevertheless, the boundary layer and flow core problems in that case must be solved simultaneously and joined at the outer edge of the boundary layer.

In paper [3], in the calculation of the variation of the potential in a boundary layer, it was found that $\varphi_\infty = \varphi_\infty(x)$, but the author did not pay attention to the fact that this is connected with the presence of an axial electric field, which should have been taken into account in formulating the problem for both the flow core and the boundary layer.

We emphasize once more that the magnitude of the axial field depends in an essential way on the rate of change of the parameters along the length of the channel (see (1.9)). If the parameters in the flow core do not change along the length of the channel and the boundary layer is nearly self-similar ($r_0 \approx$ const), then $\partial E_\tau / \partial y = 0$ in the boundary layer.

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